Warm-Up

Arithmetic Sequences — Session 1

CST: Algebra 1 (Standard 7.0)

1. Some ordered pairs for a linear function of x are given in the table below.

x	y
1	2
2	8
3	14
4	20

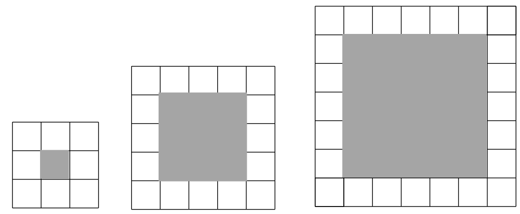
Which of the following equations can be used to generate the table above?

A. $y = x + 1$
B. $y = 3x - 2$
C. $y = 4x + 2$
D. $y = 6x - 4$

Show two different ways to find the answer.

Current: Algebra 2 (Standard 22.0)

2. Suppose the pattern of square donuts continues. Find the number of squares in the 3rd, 4th, 5th, and 6th donuts.

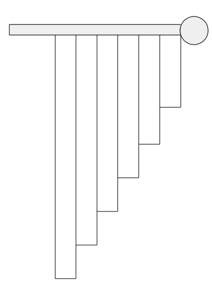


Donut (n)	1	2	3	4	5	6
Number of squares (a_n)	8	16				

Describe any patterns that you see.

Review: Algebra 2 (Standard 22.0)

3. Lisa needs to cut 6 pieces of ribbon to make a banner as shown: The first piece of ribbon is 9 inches long, and each succeeding piece has to be 4 inches longer than the previous piece. What is the length of each piece?



Other: Algebra 1 (Standard 7.0)

4. Find the slope and y-intercept for the line that contains the points $(7, 18)$ and $(13, -24)$

Arithmetic Sequences
Part 1

Over-Arching Question

Broadcasting

In radio broadcasting, autumn is one of the most critical time periods for ratings. Radio stations try to pull in as many listeners as they can by using a variety of gimmicks. A radio station had a contest in which listeners had a chance to win \$1000 every hour. In order to win, listeners needed to call in and correctly answer a contest question. The contest started with \$1000 and \$97 was added for the next caller each time the previous caller answered the question incorrectly. Suppose you were the 18th caller and the first to answer the question correctly. How much money would you win?

Ask students to read the Over-Arching Question, then briefly discuss with a partner how they might find the answer. Tell them we will revisit this later.

Definitions:
An **arithmetic sequence** is a set of numbers where the difference between each pair of successive terms is the same. We call this difference the **common difference**.

Examples: Find the common difference d for each arithmetic sequence.

1) 2, 5, 8, 11, 14, . . .

3 3 3 3

$d = 3$

When the sequence is increasing, d is positive.

2) 7, 3, -1, -5, . . .

-4 -4 -4

$d = -4$

When the sequence is decreasing, d is negative.

The terms in a sequence can be numbered a_1, a_2, a_3, …, where a_1 refers to the first term, a_2 refers to the second term, and so on. a_n is used to refer to the term in the n^{th} position.

<u>Finding terms in a sequence</u>

Example 1: Find d, the 10th term, the 200th term and the nth term.

Position	1	2	3	4	5	6	7	8	9	10	…	200	…	n
Term	7	8	9	10	11									

(a) $d = 8 - 7$
$\quad = 1$

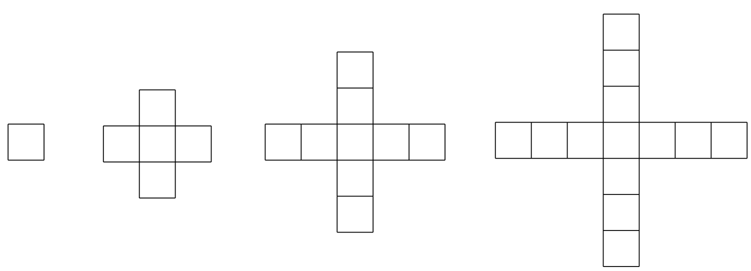
(b) To find the 10th term, continue to fill in the table using the common difference. Add 1 to each term to get the next term.
$a_{10} = 16$

(c) To find the 200th term, look for a pattern relating the position (top row) to the term (second row) Notice you can add 6 to the position to get the term.
$$a_{200} = 200 + 6$$
$$= 206$$

(d) To find the nth term, we write a rule based on the pattern we found in part (c).
$a_n = n + 6$ This rule for finding the nth term can be used to find any term in the sequence.

Finding a rule when $d \neq 1$

Example 2: The first four crosses in a pattern are shown below. If the pattern of crosses continues, what is the rule for the number of squares in the nth cross? What is the number of squares in the 200th cross?

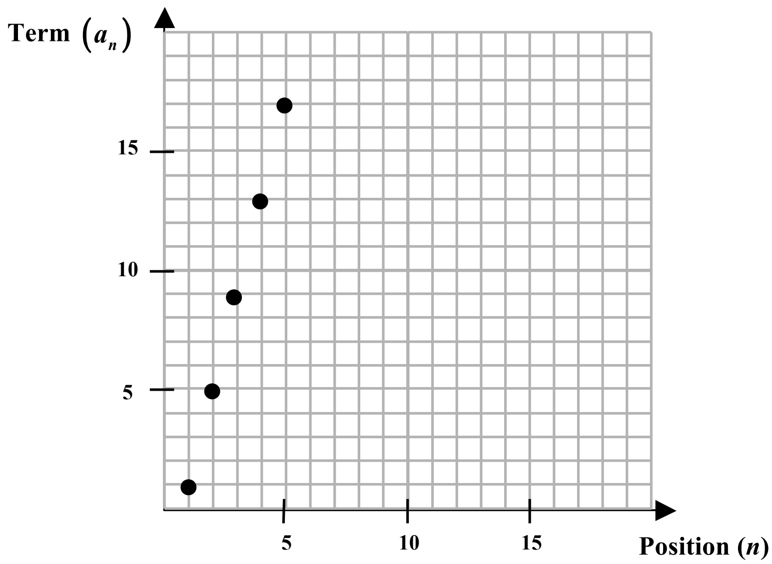


Cross (n)	1	2	3	4	5	6	…	n	…	200
Number of Squares (a_n)	1	5								

Finding the rule (method 1):

(a) Fill in the number of squares for the 3rd, 4th, 5th, and 6th crosses. (9, 13, 17, 21)
What is the common difference? ($d = 4$)

(b) Let's look for a pattern relating the position ***n*** to the term a_n by graphing the ordered pairs in the table.



(c) What do you notice about the shape of the graph? (It is linear).

(d) What is the slope? (4)

(e) Write a linear equation for the graph in slope-intercept form.
We can use d to find the y-intercept by working backwards:

Temporarily let x = the cross number and let y = the number of squares

Cross (x)	0	1	2	3	4	5	6	…	n	…	200
Number of Squares (y)	-3	1	5	9	13	17	21				

We find the y-intercept by letting x equal zero.
Working backwards, we subtract d :
$$1-4=-3$$
so the y-intercept is -3.

Using the fact that the slope = 4 and the y-intercept is -3, the equation is
$$y = 4x - 3$$

If we replace x with n and replace y with a_n, the pattern can be described by the rule:
$$a_n = 4n - 3$$

(f) Compare the slope of the line with the common difference *d*. What do you notice? (They are the same)

Discuss with students why we use $a_n = 4n - 3$ as our rule rather than the equation $y = 4x - 3$. The linear equation $y = 4x - 3$ describes a set of points where *x* can be any real number, but in the rule $a_n = 4n - 3$, *n* can only be a non-negative integer since it represents the position of a term in the sequence. For example, *n* can't be $2\frac{1}{2}$ since there is no term in the $2\frac{1}{2}$ position.

(g) Find the 200^{th} term.

$$\begin{aligned} a_n &= 4n - 3 \\ a_{200} &= 4(200) - 3 \\ &= 797 \end{aligned}$$

Finding the rule (method 2):

Multiply the top row by *d*. You can record the results in another row.

Cross (*n*)	1	2	3	4	5	6	…	*n*	…	200
Multiply *d*(*n*)	4	8	12	16	20	24				
Number of Squares (a_n)	1	5	9	13	17	21				

Look for a pattern relating these new values with the numbers in the bottom row. (You can subtract 3 from each of the numbers $d(n)$)

What is the rule? ($a_n = 4n - 3$)

Why does this work?

When we use a linear equation in slope-intercept form to find the value of *y* for any value of *x*, we first multiply by the slope, which equals *d*. When you found the number you need to add or subtract to the value of $d(x)$, you were finding the *y*-intercept.

General Rule for the n^{th} term

Position	1	2	3	4	5	6	n
Term							

Let's fill in the table. (Have students fill in the table as you lead them through a series of questions).

(a) The first term is a_1. How do we get the second term? (Add d. This gives $a_1 + d$)
(b) How do we get the third term? (Add d to the second term. This gives $a_1 + 2d$)
(c) What is the 4th term? ($a_1 + 3d$)
(d) Fill in the 5th and 6th terms.
(e) Notice for each term we add d again. How does the coefficient for d compare to the position of the term you are finding? (The coefficient of d is one less than the position.)

(f) What should we write for the n^{th} term? ($a_1 + (n-1)d$)

Position	1	2	3	4	5	6	n
Term	a_1	$a_1 + d$	$a_1 + 2d$	$a_1 + 3d$	$a_1 + 4d$	$a_1 + 5d$	$a_1 + (n-1)d$

General rule for finding the n^{th} term in an Arithmetic Sequence:

$$a_1 + (n-1)d$$

Example 3: Find a rule for the n^{th} term of the sequence 7, 18, 29, 40,… Then find a_{20}.

Method 1: Using the General Rule

We know $d = 11$ and $a_1 = 7$

$$a_n = a_1 + (n-1)d$$
$$a_n = 7 + (n-1)11$$
$$a_n = 7 + 11n - 11$$

Rule: $a_n = 11n - 4$

$$a_{20} = 11(20) - 4$$
$$= 220 - 4$$
$$= 216$$

Method 2: Using the slope and the y-intercept

The relationship for an arithmetic sequence is linear.
The slope is 11 (the common difference). We use the pattern to find the y-intercept.

Position	0	1	2	3	4
Term	–4	7	18	29	40

Rule: $a_n = 11n - 4$

$$a_{20} = 11(20) - 4$$
$$= 220 - 4$$
$$216$$

Example 4: **(You Try)** Find a rule for the n^{th} term of the sequence $5,\ 3,\ 1,\ -1,\ldots$
Then find a_{150}. Use both methods.

Think-Pair-Share

- Have students work quietly on their own for a couple of minutes. Then have them discuss their solution with a partner.
- Have each student write which method they prefer and why.
- Have the groups share out the answers and ask questions to determine the process used by the students.

Method 1: Using the General Rule

We know $d = -2$ and $a_1 = 5$

$$a_n = a_1 + (n-1)d$$
$$a_n = 5 + (n-1)(-2)$$
$$a_n = 5 - 2n + 2$$

Rule: $a_n = -2n + 7$

$$\begin{aligned} a_{150} &= -2(150) + 7 \\ &= -300 + 7 \\ &= -293 \end{aligned}$$

Method 2: Using the slope and the y-intercept

The relationship for an arithmetic sequence is linear.
The slope is -2 (the common difference). We use the pattern to find the y-intercept.

Position	0	1	2	3	4
Term	7	5	3	1	-1

Rule: $a_n = -2n + 7$

$$\begin{aligned} a_{150} &= -2(150) + 7 \\ &= -300 + 7 \\ &= -293 \end{aligned}$$

Revisit the Overarching Question:

Over-Arching Question

Broadcasting

In radio broadcasting, autumn is one of the most critical time periods for ratings. Radio stations try to pull in as many listeners as they can by using a variety of gimmicks. A radio station had a contest in which listeners had a chance to win $1000 every hour. In order to win, listeners needed to call in and correctly answer a contest question. The contest started with $1000 and $97 was added for the next caller each time the previous caller answered the question incorrectly. Suppose you were the 18th caller and the first to answer the question correctly. How much money would you win?

Is this an arithmetic sequence? If so, what is the first term and the common difference?

$a_1 = 1000$ and $d = 97$.

What do we want to find? (The eighteenth term, a_{18}.)

$$a_n = a_1 + (n-1)d$$
$$a_n = 1000 + (n-1)97$$
$$a_n = 1000 + 97n - 97$$
$$a_n = 97n + 903$$

$$a_n = 97n + 903$$
$$a_{18} = 97(18) + 903$$
$$a_{18} = 1746 + 903$$
$$a_{18} = 2649$$

I would win $2,649.

Arithmetic Sequences (part 1)
Activity

Finding terms in a sequence

Example 1: Find d, the 10th term, the 200th term and the nth term.

Position	1	2	3	4	5	6	7	8	9	10	…	200	…	n
Term	7	8	9	10	11									

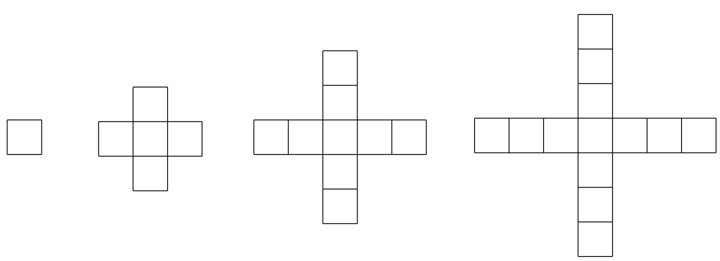
(a) Find d.

(b) Continue to fill in the table using the common difference. What is the 10th term?

(c) Find a pattern relating the position (top row) to the term (second row).
Describe the pattern, then use the pattern to find the 200th term.

(d) To find the nth term, write a rule based on the pattern we found in part (c).

Example 2: The first four crosses in a pattern are shown below. If the pattern of crosses continues, what is the rule for the number of squares in the nth cross? What is the number of squares in the 200th cross?

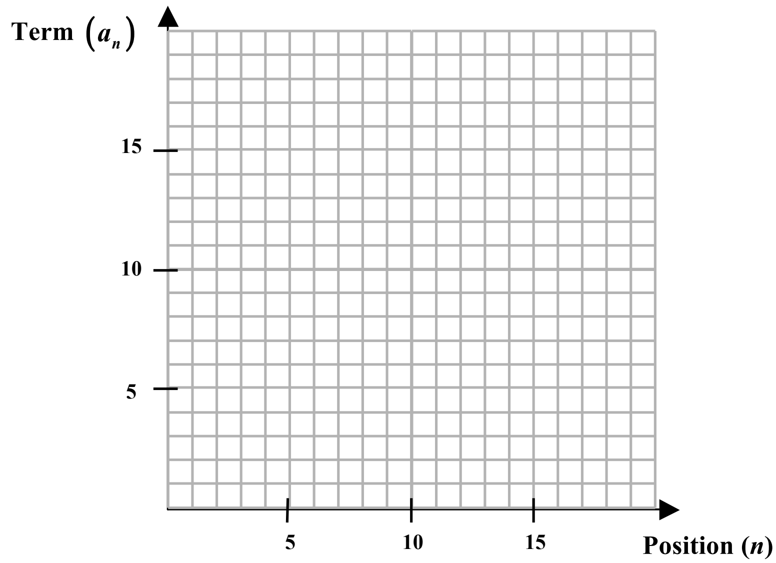


Cross (n)	1	2	3	4	5	6	…	n	…	200
Number of Squares (a_n)	1	5								

Finding the rule (method 1):

(a) Fill in the number of squares for the 3rd, 4th, 5th, and 6th crosses.
What is the common difference?

(b) Let's look for a pattern relating the position $\boldsymbol{n}$ to the term $\boldsymbol{a_n}$ by graphing the ordered pairs in the table.



(c) What do you notice about the shape of the graph?

(d) What is the slope?

(e) To write a linear equation for the graph in slope-intercept form, we can use d to find the y-intercept by working backwards:

Temporarily let x = the cross number and let y = the number of squares.

Cross (x)	0	1	2	3	4	5	6	…	n	…	200
Number of Squares (y)											

What is the y-intercept?

Write an equation of the line in slope-intercept form:

To write the rule for this sequence, replace x with n and replace y with a_n

(f) Compare the slope of the line with the common difference d. What do you notice?

(g) Find the 200^{th} term.

Finding the rule (method 2):

Multiply the top row by d. You can record the results in another row.

Cross (n)	1	2	3	4	5	6	…	n	…	200
Multiply $d(n)$										
Number of Squares (a_n)										

Describe a pattern relating these new values with the numbers in the bottom row.

What is the rule?

<u>General Rule for the n^{th} term</u>

Position	1	2	3	4	5	6	n
Term							

General rule for finding the n^{th} term in an Arithmetic Sequence:

Example 3: Find a rule for the n^{th} term of the sequence 7, 18, 29, 40,… Then find a_{20}.

Example 4: (You Try) Find a rule for the n^{th} term of the sequence $5,\ 3,\ 1,\ -1,\ldots$
Then find a_{150}. Use both methods.

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Over-Arching Question

Broadcasting

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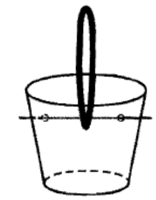
Is this an arithmetic sequence? If so, what is the first term and the common difference?

What do we want to find?

Arithmetic Sequences
Part 2

Lab Activity: Bungee Jumper

The weight of a jumper has an effect on the stretching of the bungee. You decide to create an experiment to measure the degree of stretch in relation to the weight. You will need a rubber band, a small paper cup, a paper clip, several washers, a centimeter ruler, a wooden dowel rod, and tape.



1. Straighten the paper clip and use it to punch two small holes on opposite sides of the cup about a centimeter below the rim of the cup. Loop one of the rubber bands inside the cup and slide the paper clip through the two holes and through the loop of the rubber band, so that the cup could be held by the rubber band like a bucket on a rope.

2. Tape a wooden dowel to the top of the table so that about 13 centimeters of the dowel hangs over the edge of the table. Slide the free end of the rubber band over the dowel so that the cup hangs from the dowel. Measure the distance to the nearest millimeter from the top of the cup to the base of the dowel. Record this measure in the table as your initial length with zero washers.

3. Measure the length of the rubber band when the cup contains 6 washers, 10 washers, 18 washers, and 25 washers. Record the lengths in the table:

Number of Washers	0	6	10	18	25
Length of rubber band (mm)					

After the lesson:

(a) Find the rule for determining the length of the rubber band when the cup contains n washers.

(b) Use the rule you found in part a to determine the length of the rubber band if the cup contains 16 washers

<u>**Writing a Rule for an Arithmetic Sequence given One Term and the Common Difference**</u>

Example 1: Write a rule for the nth term of an arithmetic sequence where $a_6 = -16$ and $d = 9$.

Solution: Since a_n can refer to any term, let $a_n = a_6$. This means we can replace a_n with -16 and n with 6.

Use the General Rule to find a_1, the first term:

$$a_n = a_1 + (n-1)d$$
$$-16 = a_1 + (6-1)(9)$$
$$-16 = a_1 + (5)(9)$$
$$-16 = a_1 + 45$$
$$a_1 = -61$$

Use the General Rule again to find the rule for this sequence.

$$a_n = a_1 + (n-1)d$$
$$a_n = -61 + (n-1)(9)$$
$$a_n = -61 + 9n - 9$$
$$\boxed{a_n = 9n - 70}$$

Writing a Rule for an Arithmetic Sequence given Two Terms

Example 2: Write a rule for the *n*th term of the arithmetic sequence with the 2 given terms $a_4 = 31$ and $a_{10} = 85$

Method 1

Write a system of two equations. Use the General Rule.

$$a_n = a_1 + (n-1)d$$

For equation 1, use $a_4 = 31$. $31 = a_1 + (4-1)d$

For equation 2, use $a_{10} = 85$. $85 = a_1 + (10-1)d$

Solve the system to find *d*:

$$31 = a_1 + 3d$$
$$85 = a_1 + 9d$$

$$\begin{array}{rr} & -31 = -a_1 - 3d \\ (+) & 85 = a_1 + 9d \\ \hline & 54 = 6d \\ & 9 = d \end{array}$$

Substitute the value for *d* into one of the equations to find a_1:

$$31 = a_1 + 3d$$
$$31 = a_1 + 3(9)$$
$$31 = a_1 + 27$$
$$a_1 = 4$$

Write the rule for the sequence: $a_n = 4 + (n-1)(9)$

$$a_n = 4 + 9n - 9$$
$$\boxed{a_n = 9n - 5}$$

Method 2

Use $a_4 = 31$ and $a_{10} = 85$ to write two ordered pairs, $(4,31)$ and $(10,85)$

Now find the equation of the line through these points.

Slope:

$$m = \frac{85-31}{10-4}$$
$$m = \frac{54}{6}$$
$$m = 9$$

Using point-slope form of a line and choosing one of the points gives us the following equation:

$$y - y_1 = m(x - x_1)$$
$$y - 85 = 9(x - 10)$$
$$y - 85 = 9x - 90$$
$$y = 9x - 5$$

Now write the equation as a rule for this sequence:

$$\boxed{a_n = 9n - 5}$$

Example 8: (You Try) Write a rule for the *n*th term of the arithmetic sequence with the 2 given terms:

$$a_5 = -8 \text{ and } a_{12} = -36$$

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Method 1
Write a system of two equations. Use the General Rule.

$$a_n = a_1 + (n-1)d$$

For equation 1, use $a_5 = -8$. $\quad -8 = a_1 + (5-1)d$

For equation 2, use $a_{12} = -36$. $\quad -36 = a_1 + (12-1)d$

$$-8 = a_1 + 4d$$

Solve the system to find *d*: $\quad -36 = a_1 + 11d$

$$\begin{array}{rrl} & 8 = & -a_1 - 4d \\ (+) & -36 = & a_1 + 11d \\ \hline & -28 = & 7d \\ & -4 = & d \end{array}$$

Substitute the value for *d* into one of the equations to find a_1:

$$-8 = a_1 + 4d$$
$$-8 = a_1 + 4(-4)$$
$$-8 = a_1 - 16$$
$$a_1 = 8$$

Write the rule for the sequence: $\quad a_n = 8 + (n-1)(-4)$

$$a_n = 8 - 4n + 4$$
$$\boxed{a_n = -4n + 12}$$

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Method 2
Use $a_5 = -8$ and $a_{12} = -36$ to write two ordered pairs, $(5, -8)$ and $(12, -36)$ Now find the equation of the line through these points.

Slope:

$$m = \frac{-36 - (-8)}{12 - 5}$$
$$m = \frac{-28}{7}$$
$$m = -4$$

Using point-slope form of a line and choosing one of the points gives us the following equation:

$$y - y_1 = m(x - x_1)$$
$$y - (-8) = -4(x - 5)$$
$$y + 8 = -4x + 20$$
$$y = -4x + 12$$

Now write the equation as a rule for this sequence:

$$\boxed{a_n = -4n + 12}$$

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Quick Write: Which method did you use and why?

Find someone who used the other method. Did he or she get the same answer?

Finding Arithmetic Means

Definition
The terms between any two non-consecutive terms in an arithmetic sequence are called **arithmetic means.**

Example 4: Find four arithmetic means between 18 and 78.

Solution: Consider 18 and 78 as the first and sixth terms of the sequence

18,___,___,___,___,78

Then $a_1 = 18$ and $a_6 = 78$. We need to find a_2, a_3, a_4, and a_5. If we can find *d*, then we just need to add that amount to each term to find the next term. Here are two methods for finding *d*:

Method 1:
Use the general rule to find *d*.
Let $a_n = 78$. Then *n* = 6 since 78 is the 6th term.

$$\begin{aligned} a_n &= a_1 + (n-1)d \\ 78 &= 18 + (6-1)d \\ 78 &= 18 + 5d \\ 60 &= 5d \\ 12 &= d \end{aligned}$$

Method 2:
Find the difference between 78 and 18:
$78 - 18 = 60$

We can see that we will have to add *d* five times to get from 18 to 78, so we can divide 60 by 5 to get *d*.

$$\begin{aligned} d &= 60 \div 5 \\ &= 12 \end{aligned}$$

Now find a_2, a_3, a_4, and a_5 by adding 12 to each of the previous terms:

$$\begin{aligned} a_2 &= a_1 + 12 \\ &= 18 + 12 \\ &= 30 \end{aligned} \qquad \begin{aligned} a_3 &= a_2 + 12 \\ &= 30 + 12 \\ &= 42 \end{aligned} \qquad \begin{aligned} a_4 &= a_3 + 12 \\ &= 42 + 12 \\ &= 54 \end{aligned} \qquad \begin{aligned} a_5 &= a_4 + 12 \\ &= 54 + 12 \\ &= 66 \end{aligned}$$

The four arithmetic means are 30, 42, 54, and 66.

Revisiting the Lab Activity:

After the lesson:

(a) Find the rule for determining the length of the rubber band when the cup contains *n* washers.

(b) Use the rule you found in part a to determine the length of the rubber band if the cup contains 16 washers.